Researches on polyhedra, Part I

A.-L. Cauchy

Translated into English by Guy Inchbald, © 2006

from the original:

A.-L. Cauchy, "Recherches sur les polyèdres. Première partie," *Journal de l'École Polytechnique*, **Cahier 16**, t. 9 (1813), pp. 68-74.

Translator's note

Augustin-Louis Cauchy was the first to prove that the four regular star polyhedra, described by Poinsot, are the full set. He derived them by stellating the Platonic (regular convex) solids. His proof draws on the ability to rotate a copy of a regular solid and then superimpose it exactly on the original, setting the foundations for what we would nowadays call symmetry orbits.

The *Journal de l'École Polytechnique* was organised in a confusing way. One or more consecutive Cahiers (Books) were gathered in a single Tome (sometimes translated as "Volume"), while existing bound Volumes (e.g. in the ENSAM archive at Aix-en-Provence) apparently may also gather one or more consecutive Cahiers though not necessarily corresponding to the division into Tomes.

Cauchy's writings in Cahier 16 form two consecutive but independent Memoirs, each divided into a few words of introduction followed by two main Parts. The present paper was reprinted in Série 2, Tome 1 of his Oeuvres Complètes, published in 1905 and re-published electronically by the Académie des sciences in 1995.

This translation is made from the electronic (1995) version. Only the brief introduction and Part I of the first Memoir are translated here. I have tried to translate the original fairly literally, while sometimes taking liberties in the interest of readability in the English idiom. But I am not a scholar, so please bear with my failings. In particular, rather than stick with the notion of various "polyhedra of higher kind" I tend to use the accepted English names, including "star polyhedra" in general, though in some ways these lack the self-explanatory quality of the original – possibly I am wrong to do so.

Where it seemed appropriate I have added comments in square brackets, for example where my liberties stray particularly far from the literal or where I have left a title in the original French.

Researches on polyhedra
First memoir(1)

Journal de l'École Polytechnique, Book 16, Volume 9, p.68: 1813

The memoir which I have the honour of submitting to the Class contains diverse researches on solid geometry. Part I offers the solution to the question proposed by M. [Monsieur] Poinsot, on the number of regular polyhedra which one may construct; Part II contains the demonstration of a new theorem on polyhedra in general.

Part I

M. Poinsot, in his *Mémoire sur les polygones et les polyèdres* [Memoir on polygons and polyhedra], having given the descriptions of four polyhedra of a higher kind than that which one is accustomed to consider, poses the following question: "Is it impossible for there to exist regular polyhedra whose number of faces is not one of 4, 6, 8, 12, 20? There," he adds, "is a question which merits investigation, and which does not appear to be easy to resolve with full rigour."

It is true that the diversity of methods which M. Poinsot used to derive the three new dodecahedra and the new icosahedron from the ordinary dodecahedron and icosahedron, leaves in doubt the possibility of resolving the preceding question; but, in generalising a few principles contained in the same Memoir of M. Poinsot, we succeed in deriving the regular star polyhedra [*Lit:* regular polyhedra of higher kind] from those of the first kind, by a simple analytical method which leads immediately to the solution of the proposed question.

It is easy to see, and M. Poinsot made this observation, No. 15 in his Memoir, that we may form all the regular star polygons, by extending the sides of regular convex polygons [*Lit:* regular polygons of the first kind].

The regular star polyhedra derive in an analogous manner from the regular convex polyhedra, and we may form all the new regular polyhedra by extending the edges or faces of the previously known regular polyhedra.

Thus, for example, by extending the edges which form the sides of the twelve pentagons of the ordinary dodecahedron, we obtain the small stellated dodecahedron [*Lit:* stellated dodecahedron of the second kind].

If, for the ordinary dodecahedron, we extend the plane containing each face up to the planes of the five faces which surround the opposite face, we will obtain the great dodecahedron [*Lit:* stellated dodecahedron of the third kind], understood as having the ordinary dodecahedron beneath the inner pentagons [*Lit:* pentagons of the first kind].

Finally, if we extend the edges which form the sides of the twelve pentagons of the great dodecahedron, we will obtain the great stellated dodecahedron [*Lit:* dodecahedron of the fourth kind].

We will obtain the great icosahedron [*Lit:* icosahedron of the seventh kind] by extending each face of the ordinary icosahedron until it meets the planes of the three triangles which surround the opposite face to the one being considered.

(1) Read to the first Class of the Institute, in February 1811, by A.-L. Cauchy, Civil Engineer [*Lit.* Engineer of Bridges and Roads].

What we have just seen in relation to the four star polyhedra applies generally, which is to say that we can only construct regular star polyhedra in as much as they result from extending the faces or edges of convex regular polyhedra of the same order.

Indeed, let us suppose that we have managed in some manner to construct a regular star polyhedron. We transport ourselves in thought to the centre of the inscribed sphere. The plains which comprise the different faces of the polyhedron will present, to the eye of an observer placed at this centre, the form of a convex polyhedron of the first kind, which will serve as the core of the star polyhedron. I state, further, that the regularity of the star polyhedron necessarily entails the regularity of the convex polyhedron which serves as its core.

In order to prove this, we return to the definition of regular polyhedra. A regular polyhedron of whatever kind is one which is formed by equal and regular polygons, equally inclined one to another, and joined in the same number around each vertex. It follows from this definition that, if we construct a second regular polyhedron equal to the first, and we designate the corresponding faces of the two polyhedra by the numbers 1, 2, 3, 4, ..., we can make the second polyhedron coincide with the first, by placing any one of the faces of the second on a given face, for example face No. 1, of the first, and having thus begun, by making any two of the edges of these two faces coincide. Reciprocally, if two equal polyhedra satisfy the preceding condition, we can conclude with certainty that they are regular: for, since we can then make each of the faces of the second coincide with a given face of the first, beginning by making any two edges of these two faces coincide, it follows that the different faces are equal and regular polygons; and since in making any two arbitrarily chosen faces coincide, we make all the others coincide, we conclude from this that the different dihedral angles are equal, or, which comes to the same thing, that the faces are equally inclined one to another, and joined in the same number around each vertex.

Given this, we consider a regular star polyhedron, having for its core a convex polyhedron of the same order, whose regularity has not yet been demonstrated. Construct a second star polyhedron equal to the first: at the same time you will construct a second convex polyhedron equal to that which forms the core of the given regular polyhedron; now designate, with the numbers 1, 2, 3, ..., the different corresponding faces of the two star polygons and, with the same numbers 1, 2, 3, ..., the faces of the convex polyhedra which are contained in the same planes as the faces, designated by these numbers, in the star polyhedra. In whatever manner you make the two star polyhedra coincide, the two convex polyhedra contained beneath the same faces will also coincide; and just as one can make the two regular star polyhedra coincide by placing any face of the second onto a given face of the first, it follows from this that one can in the same way make the two convex polyhedra coincide. In consequence, the different faces of the two convex polyhedra are all equal to each other, equally inclined one to another, and joined in the same number around each vertex.

It remains for us to prove that the different faces of each convex polyhedron are regular polygons. In order to achieve this it is sufficient to observe that, if in whatever manner we make a face of the second star polyhedron coincide with a given face of the first polyhedron of the same kind, the two faces having the same numbers in the convex polyhedra will also coincide: now, let us suppose that in the two regular star polyhedra the number of sides of each face is equal to n. There will be n different ways of arranging a coincidence of the two faces of these polyhedra; and consequently, there will also be *n* ways of arranging a coincidence of the corresponding faces of the two convex polyhedra. Now, we can only satisfy this condition by supposing the faces of the convex polyhedra to be equal, either to two regular polygons of order *n*, or to two semi-regular polygons of an order at least equal to $2n$; further, it is easy to see that this last case cannot exist: for, since we cannot have $n = 2$, we must have at least $2n = 6$; and, in this case, we would have convex polyhedra, all of whose faces had at least six sides, which is impossible.

Therefore it is now proved that, in any order [*Note:* "order" as in "family," not as in "sequence"], one can only construct regular star polyhedra in as much as they result from extending the edges or faces of convex regular polyhedra of the same order which form their core; and that, in each order, the faces of star polyhedra must have the same number of sides as those of convex polyhedra.

It follows [firstly] from this that, just as there are only five orders of polyhedra which provide regular convex polyhedra, we may only look among these five orders for regular star polyhedra. Thus all the regular polyhedra, of whatever kind they may be, must be tetrahedra, hexahedra, octahedra, dodecahedra or icosahedra. Further, all the tetrahedra, octahedra, and icosahedra, of whatever kind they may be, must have equilateral triangles as faces, the hexahedra squares, and the dodecahedra regular convex or star pentagons. Let us now see how many different types each order contains.

In order to shed further light [*Lit.* daylight] on this discussion, I would observe:

1st That, from the regular convex polyhedra, we can only derive regular star polyhedra by extending the edges of the existing faces, or by forming new faces;

2nd That the dodecahedron is the only regular polyhedron from which we may obtain different kinds by extending the edges of the faces, because there are two kinds of pentagon, while there is only one kind of triangle and one kind of square;

3rd That, in the case where we form new faces, we can only obtain them by extending each of the faces of the convex polyhedron until it meets the planes which contain the faces [which are] not adjacent to the one under consideration;

4th That these last must be equal in number to the faces adjacent to the one under consideration, and must all have an equal inclination both to this one and between themselves.

For the tetrahedron, each of the four faces is adjacent to three others, from which it follows that one can not obtain new faces by extending the existing ones; thus there is only one tetrahedron, that of the convex kind.

In the hexahedron, faces which are not adjacent are parallel and, consequently, can not meet: there is also thus only one hexahedron, that of the convex kind.

The ordinary octahedron may be considered as being formed by two opposite faces contained in parallel planes, each of which is adjacent to three other faces equally inclined to it and to its opposite. Thus if one hopes to form a new regular octahedron, this can only occur by extending, up to the meeting point of the faces, the planes which contain the three faces adjacent to the opposite one: now this construction, instead of giving a regular star octahedron, gives a double solid formed by two tetrahedra which mutually intersect. This is just like when, extending the sides of the ordinary hexagon, we obtain two equilateral triangles crossing each other instead of a star hexagon.

If, for the ordinary dodecahedron, we extend the sides of the twelve pentagons, we will have, just as M. Poinsot observed, a regular small stellated dodecahedron.

To obtain the other dodecahedra, it is necessary to find the means to extend, until it meets each face of the ordinary dodecahedron, five faces [which are] not adjacent but are equally inclined to it. Now, the ordinary dodecahedron can be considered as being formed by two opposing faces lying in parallel planes, and thus each is adjacent to five other faces equally inclined to it and to its opposite. Therefore if we can construct other dodecahedra than those described above, this can only be by extending each face of the ordinary dodecahedron until it meets the planes which contain the five neighbours of the opposite face. The intersections of these five planes with the face under consideration form two regular pentagons, one of the convex and one of the star type. These two pentagons represent the faces of the great and great stellated regular dodecahedra.

For the ordinary icosahedron, in choosing an arbitrary face for the base, we find, as with the three preceding orders, another face lying opposite and in a parallel plane. If we classify the triangles contained between these two faces according to series, placing those which are equally inclined to the base within the same series, or, which comes to the same thing, to the opposite face, we will find that the eighteen remaining triangles form four series, to wit:

1st One series of three triangles, adjacent to the base;

2nd One series of three triangles, adjacent to the opposite face;

3rd One series of six triangles, of which each has only one vertex shared with the base;

4th One series of six triangles, of which each has only one vertex shared with the face opposite.

Let us designate the triangles in the third and fourth series by the numbers 1, 2, 3, 4, 5, 6; such that two consecutive numbers indicate two triangles which touch at an edge or a vertex. The base of a new regular icosahedron can only be formed by the intersection of the base of the given icosahedron with three triangles in the same series, equally inclined one to another. Given this, it is easy to see that we can only hope to obtain the base of a new icosahedron in five ways, to wit, by extending, up to the meeting point of the plane of the given base:

1st The planes which contain the three triangles of the second series;

2nd The planes which contain triangles 1, 3, 5 of the third series;

3rd The planes which contain triangles 2, 4, 6 of the third series;

4th The planes which contain triangles 1, 3, 5 of the fourth series;

5th The planes which contain triangles 2, 4, 6 of the fourth series.

If we apply the five preceding constructions step by step to the different faces of the ordinary icosahedron, we will obtain the following results:

1st By following the first construction, we will traverse all the faces, and we will obtain the great icosahedron described by M. Poinsot;

2nd By following the second or third construction, we will only traverse eight faces, and we will simply obtain a convex regular octahedron;

3rd By following the fourth or fifth construction, we will only traverse four faces, and we will simply form a regular tetrahedron.

It follows from what has just been said, that we can only form the four other regular star polyhedra described by M. Poinsot.

The preceding theory further furnishes the means to calculate the angle contained between any two faces of a regular polyhedron, where we know the angles formed by adjacent faces in the convex tetrahedron, dodecahedron and icosahedron.

In fact, these three angles being α, β, γ;

The angle contained between two faces of the tetrahedron will always be α.

For the hexahedron, two adjacent faces interest at a right angle, two non-adjacent faces are parallel.

In the octahedron, the faces are in parallel pairs. The angle contained between two non-parallel faces is represented by

$$\pi - \alpha$$

when the two faces are adjacent, and by α when they are not.

In the dodecahedron, the faces are in parallel pairs. The angle contained between two non-parallel faces is represented by β when the two faces are adjacent, and by

$$\pi - \beta$$

when they are not.

In the icosahedron, the faces are also in parallel pairs. The angle contained between one face and the adjacent faces being γ, the angle contained between the given face and the faces adjacent to the opposite face will be

$$\pi - \gamma;$$

finally, the angle contained between two faces, of which one is adjacent neither to the other nor to the opposite face, will be represented either by α or by

$$\pi - \alpha.$$