Note on the theory of regular polyhedra

J. Bertrand

Translated into English by Guy Inchbald, © 2006

from the original:

J. Bertrand, "Note sur la théorie des polyèdres réguliers", *Comptes rendus des séances de l'Académie des Sciences* **46** (1858) pp. 79-82.
With further remarks, p 117.

Translator's note.

Bertrand's note is the earliest treatment I know, of the process that we now call facetting a polyhedron. We nowadays understand it as exactly reciprocal to the process of stellation, by which Kepler first derived two of the regular star polyhedra.

I have tried to translate the original fairly literally, while sometimes taking liberties in the interest of readability in the English idiom (for example I translate "on" as "we"). But I am not a scholar, so please bear with my failings.

Where appropriate I have added comments in square brackets, for example where my liberties stray particularly far from the literal or where I have left a title in the original French.

Note, also here translated (and text corrected):
ERRATA. — Page 79, lines 15 and 22, for Gourgeon, read Gourjon.

Remarks by Branko Grünbaum.

Lemma 1 is – like so many other obvious statements – wrong. First counterexample: take of all his points in one plane. Second counterexample: take the four vertices of a tetrahedron together with the four incenters of the faces. But if it were formulated correctly, it would have been sufficient for the application he makes.

In the proof of Theorem II, the statement "... this polygon is the only possible face ..." is wrong, or at least misleading – as appears in his arguments that follow.

"... the three conditions which make up the definition of a regular polyhedron ..." are succinctly formulated by Poinsot in his earlier paper, and repeated by Cauchy in his paper on regular polyhedra as follows: "A regular polyhedron ... is formed by equal and regular polygons, equally inclined to each other, and meeting in the same number at each vertex." This definition has two shortcomings: the less serious is that only pairs of faces that share an edge are meant, while the more serious is that they never define what is a "polyhedron".

His comment about Kepler is not justified. Kepler does consider the faces of his polyhedra to be pentagrams.

Note on the theory of regular polyhedra

by **J. Bertrand**

Since the attention of the academy is drawn to the most interesting theory of polyhedra, I am taking this occasion to announce that M. [Monsieur] Gourjon, with whose skill and ingenious spirit the natural philosophers are familiar, has kindly constructed, for my pleasure, the regular star polyhedra described in Volume II of the *Memoires des Savants étrangers* [Memoirs of the foreign scholars]. These solids already existed, it is true, at M. Poinsot's, who discovered them; but, despite the kindness with which the illustrious geometer received those who wished to study them, these models were not on public display: the solids constructed by M. Gourjon will be so entirely, for they are now on display at the College de France. We know that M. Poinsot's four solids are, together with the five regular polyhedra known in antiquity, the only regular bodies whose existence is possible. M. Cauchy proved this in a Memoir presented to the academy in 1812. But his demonstration, while rigorous, requires enormous concentration and can only be followed with great self-discipline in order to verify all his assertions on the models raised on [*Lit.* in relief of] the regular convex dodecahedron and icosahedron. I shall propose a demonstration which seems to me to be easier.

Lemma I. — Given a general set of points in space, we can always find a convex polyhedron whose vertices may be taken from among the given points, and which contains all the remaining points in its interior. We do not develop the demonstration of this lemma, which is fairly obvious.

Lemma II. — There is no convex polyhedron having more than five faces meeting at every vertex.

This proposition, easily confirmed by Euler's famous theorem, has been known for a long time.

Theorem I. — A regular polyhedron, of whatever kind, necessarily has the same vertices as some regular convex polyhedron.

We know that the vertices of a regular polyhedron lie on a sphere, and every convex polyhedron whose vertices are among these points cannot in consequence contain the others in its interior; we conclude from this, by virtue of lemma I, that some convex polyhedron exists which has as its vertices every vertex of the regular polyhedron being considered.

It remains to prove that this polyhedron is regular. To achieve this, let us consider two figures P and Q equal to one another and each formed by the regular polyhedron under consideration together with the convex polyhedron having the same vertices; Not only can P, as we might suppose, be superimposed on Q, but the coincidence may be established by placing an arbitrary vertex of Q on some given vertex of P. Further, two vertices being one upon the other, the coincidence of the two regular polyhedra which form parts of P and of Q, and by extending this of the whole figures, can be done in at lest three ways, because at least three faces of the regular polyhedra meet at the vertices under consideration, and the coincidence can be established by placing some face of the second onto a given face of the first. The two solid angles of our convex polyhedra are thus not only equal but are also able to coincide in three different alignments. Now, by virtue of lemma II, these solid bodies [typo for "angles"?] are trihedral, tetrahedral or pentahedral, and in each of these three cases the threefold coincidence is impossible if they do not have faces which are both equal and equally inclined; all the faces which meet at a certain vertex of the convex polyhedron can thus be superimposed, and just as the coincidences of the two convex polyhedra may be formed by placing an arbitrary vertex of one onto a given vertex of the other, a particular face may coincide with the one identical to it, and in such manner that two arbitrary vertices are one upon the other. We conclude from this that the faces are regular polygons, and consequently the convex polyhedron fulfils the three conditions which make up the definition of a regular polyhedron, and the theorem is demonstrated.

Theorem II. — There are only four non-convex polyhedra [Lit. of higher kind]. By virtue of the previous theorem, to obtain the non-convex regular polyhedra it is evidently necessary to take the regular convex polyhedra and proceed in the following manner. Choose a vertex on one of these polyhedra and see if other vertices exist which, joined to this one, may form a regular polygon; this polygon is the only possible face for the non-convex polyhedron having the same vertices as the original. The number of congruent polygons to which a single vertex can belong will be the number of faces which surround a solid angle of the new polyhedron.

It is clear that this construction applied to the tetrahedron yields nothing.

Each vertex of the octahedron belongs to two squares, which can evidently not form the faces of a polyhedron.

Each vertex of the cube can form, with two other vertices properly chosen, an equilateral triangle, and this in three different ways, but these three triangles belong to a regular tetrahedron.

Each vertex of the regular dodecahedron can, in three different ways, form equilateral triangles having some vertices belonging to two faces which meet there, but the triangles do not make a polyhedral angle, two of them never having a common edge.

Each vertex of the regular dodecahedron may equally be considered as the vertex of six equilateral triangles whose other vertices belong to faces connected to those which contain the given vertex. But these six equilateral triangles are faces of two regular tetrahedra.

Each vertex of the dodecahedron is lastly the vertex common to three regular pentagons whose four other vertices belong to the same polyhedron. These three pentagons do not form faces of a trihedral angle, because two among them have no common edge, but the star pentagons which have the same vertices do form a trihedral angle, and their complete set, for the whole polyhedron, forms the great stellated dodecahedron [*Lit.* the regular dodecahedron of the fourth kind].

Each vertex of the icosahedron is the vertex common to five equilateral triangles having as sides the shortest lines that one can draw between the vertices beyond those which form the sides of the faces. These triangles form the great icosahedron [*Lit.* the icosahedron of the seventh kind].

Each vertex of the icosahedron may be considered as the common vertex of five convex regular pentagons, whose four other vertices belong equally to the icosahedron; these pentagons are the faces of the great dodecahedron [*Lit.* the dodecahedron of the third kind]. Finally the same vertices may be considered as belonging to the star pentagons which form the small stellated dodecahedron [*Lit.* the stellated dodecahedron of the second kind].

Therefore there are only four star polyhedra in all, being precisely those which M. Poinsot discovered.

M. Poinsot, in his 1809 Memoir, indicated with good probability the non-existence of any regular solids other than those he described. "If there existed, for example," said M. Poinsot, "a new regular polyhedron having 28 faces, and if we mark the centres of these faces, we would have an equal number of points distributed regularly over the sphere. Now by treating all these points as vertices we could make a fully convex polyhedron following the ordinary definition.... But we do not see why the polyhedron whose vertices are uniformly distributed on the sphere, should not be a perfectly regular polyhedron; we would thus have a regular convex polyhedron, the number of whose vertices was not one of the numbers 4, 6, 8, 12, 20, which has been shown to be quite impossible." (*Journal de l'Ecole Polytechnique*, **Vol. 10**, Page 43.)

M. Poinsot therefore saw clearly, without affirming in a formal manner, that every non-convex regular polyhedron was successively bound to a convex regular polyhedron. M. Cauchy proved the correctness of this assertion by taking as the polyhedron conjugate to a given polyhedron the convex core formed by its face planes. I have just shown that the convex polyhedron having the same vertices as a regular star polyhedron is necessarily regular: a quite similar demonstration would allow us to rigorously establish the exact same proposition stated by M. Poinsot, and to demonstrate that the centres of the faces of a regular polyhedron also form a regular polyhedron, and it would not be difficult to deduce from this a third proof of the theorem which is the subject of this note.

Remarks on the part incorrectly attributed to Kepler in the discovery of the four regular non-convex polyhedra.

M. Bertrand replies to an objection which was addressed to him on the occasion of his last communication. He has been reproached for having attributed the discovery of the four regular convex polyhedra to M. Poinsot. Two of these polyhedra are depicted and described in previous works; that is perfectly correct. We can see in Kepler's *Harmonices mundi*, page 182 (1), a well-executed drawing of the small stellated dodecahedron; but Kepler and the authors who spoke of these polyhedra, prior to M. Poinsot, never suspected that they were regular: they considered them as formed by sixty triangular faces and not by twelve regular pentagons; they have no more right to be cited in the story of the discovery of the regular polyhedra, than Tobie Mayer and Bradley have of the discovery of Uranus, which they had observed long before Herschel, but had taken for a star.