Polytopes - abstract and real

Norman Johnson

Edited by Guy Inchbald

Editor's preface

A "polytope" is a generalised figure in n dimensions, including polygons in two dimensions, polyhedra in three, polycells or polychora in four, and so on. Abstract polytopes are an important modern development. Based in part on this theoretical background, Johnson has developed the idea of real polytopes, as a significant step towards reconciling the inconsistencies between abstract polytopes, our intuitive ideas of what polygons and polyhedra are (and what polytopes might be), and established geometric ideas about lines, planes and other figures as sets of points in space.

This note began life as a series of messages from Norman Johnson, during conversations on a mailing list between 12 May 2003 and 18 April 2004. Although his theory was still a little rough around the edges, these messages provided the first available description both of his theory of real polytopes and of his improved formulation of the underlying abstract polytopes. I felt it worthwhile to gather the material into a coherent form, and this is the result.

Others on the list contributed occasional ideas and corrections, including Wendy Krieger, Don Hatch, Jonathan Bowers, myself, Christine Tuveson, John Conway and Jeff Tupper. These were for the most part absorbed into Norman's replies, though I have nonetheless had to do a good deal of cutting, stitching and tidying to create a readable narrative. Norman has also made a few corrections and improvements to my draft. Beyond this I have touched his words as little as possible; I have even left his diagrams in their original "ASCII art" state, not least as a reminder of this note's origin.

Guy Inchbald, February 2008.

Introduction

How one defines a polytope is a subjective matter: what we can define polytopes to be is more or less a case of what we are prepared to deal with. But it does not follow that it is futile to try to formalize one's notions. Of course, different notions require different definitions, and the lack of a standard definition may have indirectly contributed to the present richness of polytope studies. Nevertheless, "polytope" is not a primitive concept that ought to be left undefined.

Euclid shouldn't have defined "point," "line," or "straight," but he did. ("A *point* is that which has no part." "A *line* is breadthless length." "A *straight* line is a line which lies evenly with the points on itself.") On the other hand, Euclid should have defined "polyhedron," but he didn't.

Branko Grünbaum has called it the "original sin" in the theory of polyhedra that, going back to Euclid, geometers have freely discussed regularity and other polyhedral properties without ever saying what a polyhedron is. His own definitions of polygons and polyhedra provide the desired rigor but open the doors to figures that most subscribers to this List would probably find too peculiar to accept as legitimate polygons or polyhedra. For example, what looks like a triangle ABC may actually be a hexagon ABCDEF with three pairs of coincident vertices -- A and D, B and E, C and F.

It is not too hard to formulate a satisfactory definition of a convex polytope. Extensions to spheroidal and toroidal polytopes are likewise fairly straightforward. But as soon as we choose to admit star polytopes, we need to be careful. Even the great Ludwig Schläfli, who pioneered the study of higher-dimensional polytopes, refused to recognize the validity of two of the four Kepler-Poinsot polyhedra and six of the ten regular star polychora, because they do not satisfy Euler's Formula. In his book of that name, Coxeter gave appropriate definitions for regular polytopes, which apply to both convex and starry figures. But less symmetric figures, even those with as much symmetry as uniform polytopes, may have features that raise new questions and call for a refined definition.

Abstract polytopes

The important concept of "abstract polytopes" was developed by Danzer & Schulte (1982, pp. 296-298), McMullen (1989, pp. 39-40; 1993, pp. 100-102), and McMullen & Schulte (2002, pp. 22-31). This makes it possible to separate the combinatorial properties of a polytope from the geometric properties associated with its being embedded in some space.

Ludwig Danzer, Egon Schulte, and Peter McMullen define an abstract n-polytope as a partially ordered set of elements or "j-faces," subject to conditions that ensure that it is monal, so that no two j-faces coincide, that it is dyadic, so that a 1-face (edge) joins just two 0-faces (vertices) and just two (n-1)-faces (facets) meet at any (n-2)-face (ridge), and that it is properly connected, so that it does not split into a compound of two or more n-polytopes.

The properties that define an abstract polytope are combinatorial in nature. The incidence of elements of a polytope is thus an abstract property, independent of how it is realized geometrically.

The properties that determine whether and how a given abstract n-polytope can be realized as a geometric figure in Euclidean (or non- Euclidean) n-space involve the interpretation of its j-faces as points, line segments, and other suitable point sets.

The definition of an abstract polytope given by Peter McMullen and Egon Schulte in their recent book Abstract Regular Polytopes is spread over several paragraphs, interrupted by definitions and preliminary results. My definition of an abstract n-polytope is stated differently but is entirely equivalent to the definitions given by Danzer, Schulte, and

McMullen. It is a more concise, but logically equivalent, restatement of the definition in recursive form and makes a distinction between j-faces and j-facials. The word "j-facial" is my own invention, but the notion itself is implicit in the Danzer-Schulte-McMullen formulation.
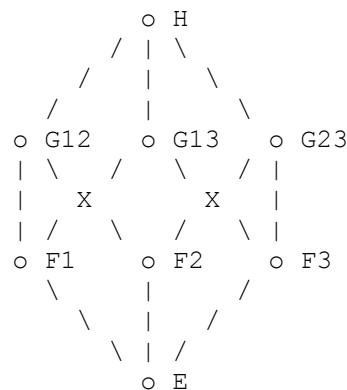
The "j-faces" of an n-polytope are its elements of rank j. Its "j-facials" are j-polytopes comprising a given j-face and all the elements of lower rank incident with it: a "j-facial" <J> is a j-subpolytope whose elements are all the i-faces incident with J for i <= j.

Thus I regard a triangle as a 2-polytope whose elements consist of one (-1)-face, 3 0-faces, 3 1-faces, and one 2-face. Geometrically, the (-1)-face or "nullity" is the empty set, the 0-faces or "vertices" are points, the 1-faces or "sides" are line segments, and the 2-face or "body" is a planar region. None of these are polytopes. A 1-facial, which *is* a polytope, is a partially ordered set consisting of a particular 1-face (side), the two 0-faces (vertices) incident with it, and the (-1)-face (empty set).

In many treatments the face/facial distinction is not strictly maintained, and elements are conflated with subpolytopes. For the most part, this does no great harm, as each j-face corresponds to a unique j-facial. But in discussing geometric polytopes with self-intersections the distinction is quite useful. One can say that the sides of a convex polygon are pairwise disjoint, while those of a star polygon are not, but only if the sides are understood as not including their end points. The question of coincident elements can be addressed in the same manner.

The monal, dyadic, and properly connected properties (also my terminology) are all exhibited in the Hasse diagram of an abstract n-polytope, which consists of nodes (representing j-faces) at n+2 levels (corresponding to ranks -1 to n), with nodes on consecutive levels joined by a branch when the associated j-faces are incident.

For example, a 2-polytope <H> = {E; F1, F2, F3; G12, G13, G23; H} is represented by the following diagram:

```
                    o H
                  / | \
                /   |   \
              /     |     \
            /       |       \
        o G12     o G13     o G23
        | \   /     \   /   |
        |   X         X     |
        | /   \     /   \   |
        o F1      o F2      o F3
          \        |        /
            \      |      /
              \    |    /
                o E
```

The polytope, which can be realized as a triangle, has one (-1) face E; three 0-faces F1, F2, F3; three 1-faces G12, G13, G23; and one 2-face H. It also has one (-1)-facial <E>; three 0-facials <F1>, <F2>, <F3>; three 1-facials <G12>, <G13>, <G23>; and one 2-facial <H>. An n-polytope also has "sections" of various ranks. For example, the abstract 1-polytope H/F1 = {F1; G12, G13; H} is a section of rank 1, the cofacial of F1.

The difference between j-faces and j-facials is that a j-face is represented by a single node, e.g., G12, while a j-facial is the Hasse subdiagram topped by a given node, e.g. <G12> = {E; F1, F2; G12}. The definition of an abstract n-polytope includes a completeness property: there is just one (-1)-face and just one n-face. In the diagram this means that there is just one bottom node and just one top node.

The monal property is the abstract property that relates each j-face of an n-polytope to a unique j-facial (its "span"), as well as to a unique j-cofacial (its "cospan") that includes the j-face and all the k-faces incident with it for k > j. In the diagram this means that no node can be the top (or bottom) of two different subdiagrams each representing a valid j-facial (or j-cofacial). In other words, one node cannot be superimposed on another.

The dyadic property says that every 1-face is incident with just two 0-faces and every (n-2)-face is incident with just two (n-1)-faces. In the diagram this means that if there is a path joining a pair of nodes I and K two levels apart, then there are just two such paths, {I; J1; K} and {I; J2; K}.

The property of proper connectedness says that no proper subset of the j-faces has all the previous properties. In the diagram this means that no proper subset of nodes and branches is itself the Hasse diagram of a valid n-polytope. This implies that neither the polytope nor any of its sections splits.

If we stipulate that a geometric n-polytope must first of all qualify as an abstract n-polytope, then we require that it be properly connected. This rules out polytopes with compound vertex figures and the like.

Realizing abstract polytopes

The definition of an abstract polytope is not to be tinkered with, but abstract polytopes can be realized as geometric figures in a variety of ways. Which of the many possible realizations are deemed acceptable depends entirely on what objectives one has in mind. In some investigations only convex figures may be relevant. In other cases, such as Grünbaum's "Polyhedra with hollow faces" (1993), vertices and edges may coincide.

Branko and I are in agreement both with each other and with Danzer, Schulte, and McMullen (whom I have also cited) as to what constitutes an abstract polytope. We are also in agreement that there are many ways of realizing an abstract polytope as a geometric figure and that there is no one "right" way. The combinatorial structure of a geometric polytope is an abstract property. Other properties, such as edge lengths, angles, and symmetry, depend on the space in which the polytope is realized and on how abstract "j-faces" are represented as geometric entities.

A realization of an abstract polytope P is a mapping P --> P' taking each j-face of P into an entity (i.e., some set of points) that can be construed as an element (a vertex, an edge, etc.) of a geometric polytope P' in some space, provided that incident abstract elements of P are mapped to incident geometric elements of P'. It is necessary to have some criteria for the incidence of geometric elements; e.g., it would be reasonable to regard a vertex (point) as incident with an edge (line segment) only if the vertex is one of the edge's end points.

A realization P --> P' is *faithful* if the correspondence between abstract elements of P and geometric elements of P' is one to one. In a faithful realization elements of P' must be distinct geometric entities. This rules out coincident vertices and the like but does not prohibit intersecting or overlapping edges, faces, etc.

If a realization is not faithful then it is *degenerate*. Grünbaum does not wish to restrict himself to faithful realizations. The (mainly Euclidean) geometric polytopes he has described may well have coincident elements, even to the extreme of every 0-face of an abstract polytope being mapped to the same geometric point. Branko does not dispute that his doubly wound regular polygon {6/2} is not a faithful realization of an abstract hexagon. (He does object to such figures being termed "degenerate"; I have proposed calling them "reductive.") Because his {6/2} "looks like" a triangle {3}, it is necessary to provide labels to indicate the order in which the vertices are visited.

Grünbaum's figures are genuine geometric realizations of abstract polytopes. He is quite specific as to what conditions his geometric polytopes must satisfy, and the theory underlying the study of these figures is perfectly sound. The important thing is that whatever class of "polytopes" you care to investigate, you should be clear about the rules you are playing by.

Because of the efforts of people like Branko Grünbaum, considerable rigor has been introduced into the once hazy notions of polyhedra and polytopes. Admittedly, not all the confusion has been dispelled. For readers who are concerned about definitions, let me suggest a few useful principles.

(1) Any geometric figure that is to be called a "polytope" should have the combinatorial structure of an abstract polytope. That is, a geometric polytope should be a realization of some abstract polytope. (Even here there are exceptions. G. C. Shephard's "regular complex polytopes," not generally being dyadic, are not realizations of abstract polytopes in the Danzer-Schulte-McMullen sense.)

(2) Whether a geometric polytope should be a *faithful* realization of an abstract polytope is a matter of taste. Do you want to allow coincident vertices or not? Must your figures be labeled to be correctly understood?

(3) Any other geometric conditions should be clearly spelled out. Do you want to allow digons? What about dihedra and hosohedra? Do you want to exclude protruding whiskers and interlocked faces? Can adjacent faces be coplanar?

(4) In any case, if you claim to have a different way to define a polytope, say what you mean.

Grünbaum wants to entertain the possibility of two j-faces having the same image (coincident vertices and edges) or of j-faces having no image (polyhedra with "hollow" faces). One might also want to allow skew polygons and skew polyhedra or other generalisations. These are all perfectly legitimate alternatives but are not what I had in mind. My goal was to formulate a definition of a "real" polytope that includes, for example, all the uniform polyhedra described in the 1954 paper of Coxeter, Longuet-Higgins, and Miller but does not force one to accept anomalous figures like Skilling's "polyhedron" with twelve faces at each vertex, let alone Grünbaum's generalisations.

I prefer to concentrate on faithful realizations that have a few additional geometric properties to exclude certain anomalous cases. These include all the classical uniform and co-uniform polytopes and honeycombs in spherical, Euclidean, hyperbolic, and elliptic space. Without implying that no other figures deserve recognition, I call these "real" polytopes and honeycombs. My chief interest lies with uniform figures, and this motivated the provisions I built into my definition of a "real" polytope or honeycomb.

The definition of a "real" n-polytope has to be carefully framed if we want it to include those figures that conform to our notions of what a polytope should be while excluding those figures that do not. Our notions may prove to be incompatible or have unintended consequences and so may need to be revised. We may well want to go beyond the limitations of convex figures but may not want to go as far as Grünbaum in allowing polygons to have coincident vertices or allowing the faces of a polyhedron to be "hollow," without interiors.

My approach is based on the idea that a realization of an abstract polytope is a mapping of abstract j-faces into real points, line segments, and other geometric objects. I define certain sets of points that I call *entities*, which can be assembled in appropriate ways to form

real polytopes or honeycombs ("improper" polytopes). Each entity is an open region or union of open regions of some n-space. An entity has both a dimension and a rank and may be characterized as either *proper* or *improper*. The rank of a proper entity is the same as its dimension; the rank of an improper entity is one more than its dimension. Improper entities (e.g. an entire n-space) are of interest primarily in connection with honeycombs.

Note that I only use the terms "proper" and "improper" in discussing my two kinds of entities. This usage is parallel to the notion of proper and improper integrals in calculus. The term "improper" is in no way pejorative, and my usage does not apply to anyone else's figures.

An improper entity is relevant only to honeycombs. An entire n-space is one example of an improper entity. The categories of proper and improper entities are not quite mutually exclusive. A hyperbolic line, which can be a side of an asymptotic polygon, is a proper entity of dimension and rank 1. But as an entire 1-space, a hyperbolic line can be the scope of a partition (1-dimensional honeycomb) and is thus an improper entity of dimension 1 and rank 2.

A *region* of n-space is a set of points containing a connected open set and contained in its closure. A region may be open, closed, or somewhere in between. An open region is a connected open set. The empty set counts as an open region of any n-space. A *proper convex* region is a convex region of n-space whose closure contains (0) no antipodal points, (1) at most a countable number of lines, and (2) no branch of an equidistant curve. (Antipodal points occur only in spherical space; equidistant curves occur only in hyperbolic space.) In a Euclidean n-space any *bounded* convex region is a proper convex region.

The empty set is a proper entity of dimension and rank -1. A point is a proper entity of dimension and rank 0. For n > 0, a proper entity of dimension and rank n is an open region or union of open regions of an n-space, having a connected closure that is contained in a proper convex region, an exterior (the complement of its closure) that is connected if n > 1, and a boundary that is the union of a countable number of proper j-dimensional entities (0 <= j <= n-1).

In the realization of an abstract polytope as a geometric figure, each j-face is represented by an "entity" of rank j. By the dimension of an entity I mean the dimension of the subspace in which the entity is an open set. For a proper entity the rank is the same as the dimension. A real polytope involves only proper entities. An "improper" entity of dimension j-1 and rank j is an open set in a (j-1)-dimensional space that functions as a j-face of a real honeycomb. For instance, a plane tessellation like {4, 4} has for its elements the empty set plus infinitely many vertices, edges, and faces but no *body*. Instead it has a *scope*, the entire plane, which counts as its only 3-face (an improper entity of dimension 2 and rank 3).

Entities do not have to be disjoint, but they cannot coincide. This is not true of degenerate realizations like those favored by Grünbaum. Thus monality is an abstract property that implies the geometric property of noncoincidence of elements in a faithful realization.

Real polytopes

All of the foregoing provides the necessary foundation for the following

DEFINITION. A real n-polytope P is an abstract n-polytope whose j-faces are proper entities of rank j in some real Euclidean or non-Euclidean n-space and whose j-facials of lower rank are real j-polytopes. The (-1)-face of P is the empty set, and the n-face of P is a proper entity of rank n, its body. The body is disjoint from each of the other j-faces, but its closure has a nonempty intersection with each j-face for j > -1, and its boundary is contained

in the union of the j-faces for j < n. Moreover, for -1 < j < n, all the elements common to any pair of j-facials lie in a single (j-1)-space.

Note some of the implications of this definition. A real polygon is an abstract 2-polytope whose j-faces are real entities. The (-1)-face, or *nullity*, is the empty set; each 0-face, or *vertex*, is a point; each 1-face, or *side*, is an open interval of a line; and the 2-face, or *body*, is one or more open regions of a plane. The vertices of a polygon are thus distinct points, and its sides are distinct (but not necessarily disjoint) segments the ends of which are vertices. The body is an open set with a connected closure consisting of one or more regions bounded by the vertices and sides. The rest of the plane, the exterior of the polygon's body, is a connected open set.
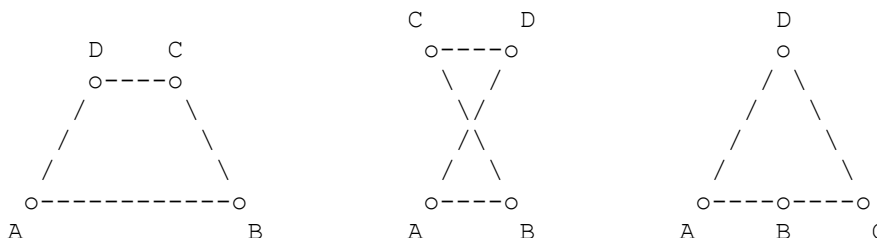
Consider the figure

```
                C
                o
               / \
              /   \
             /     \
            /       \
           o---------o
           A         B
```
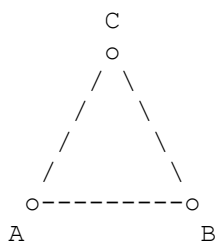
This familiar 2-polytope has as elements

   one   (-1)-face, the empty set @;

   three 0-faces, or vertices, the points A, B, C;

   three 1-faces, or sides, the line segments AB, BC, CA;

   one   2-face, or body, the plane region ABC.

It likewise has as subpolytopes

   one   (-1)-facial, the nullitope or (-1)-polytope
          <@> = {@};

   three 0-facials, the monons (0-polytopes)
          <A> = {@; A}, <B> = {@, B}, <C> = {@; C};

   three 1-facials, the ditels (1-polytopes)
          <AB> = {@; A, B; AB}, <BC> = {@; B, C; BC}, <CA> = {@; C, A; CA};

   one   2-facial, the triangle
          <ABC> = {@; A, B, C; AB, BC, CA; ABC}.

All of the figures below qualify as real quadrangles:

```
     D    C              C       D              D
     o----o              o----o                o
    /      \              \   /                / \
   /        \              \ /                /   \
  /          \              X                /     \
 /            \            / \              /       \
o--------------o          o----o           o----o----o
A              B          A    B           A    B    C
```

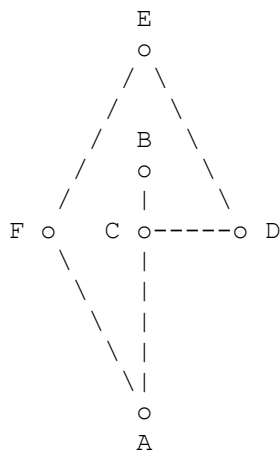(Nothing prohibits two sides of a polygon from intersecting or says that three vertices cannot be collinear.)

A crucial provision in my definition of a real n-polytope is that the closure of the body of a polytope is the union of all the elements, and its boundary is the union of the elements of lower rank. Thus the body of a triangle <ABC> is the open region ABC, whose closure consists of the region ABC together with the segments AB, BC, and CA plus the points A, B, and C, and we can always throw in the empty set.

It is this provision that prevents the figure

```
                    D
                    o
                   / \
                  /   \
                 /     \
                /       \
               o--------o------o
               A        C      B
```
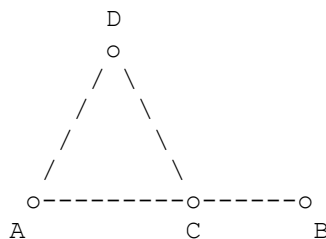
from being a valid polygon. Here the vertices are the points A, B, C, and D; the sides are the segments AB, BC, CD, and DA; and the body can only be the open region ACD. The trouble is not that sides AB and BC overlap but that vertex B and side BC are not part of the boundary of the body. The closure of the region ACD consists of the body ACD together with the sides CD and DA and part of the side AB, the vertices A, C, and D, and the empty set. The vertex B and the side BC are not included in the boundary. In general, a real polygon may not have a protruding "whisker," and a real polyhedron may not have a protruding "membrane."

Some of a polygon's vertices, and even entire sides, can be hidden inside its body. For example, the figure at the right is a legitimate hexagon. In this case the body is a single region formed by the interior of the concave pentagon ACDEF with a slit formed by the points B and C and the segment BC. The boundary of the body is the union of the six vertices and the six sides.

```
            E
            o
           / \
          /   \
         /  B  \
        /   o   \
       /    |    \
   F o    C o-----o D
      \     |
       \    |
        \   |
         \  |
          \ |
           o
           A
```
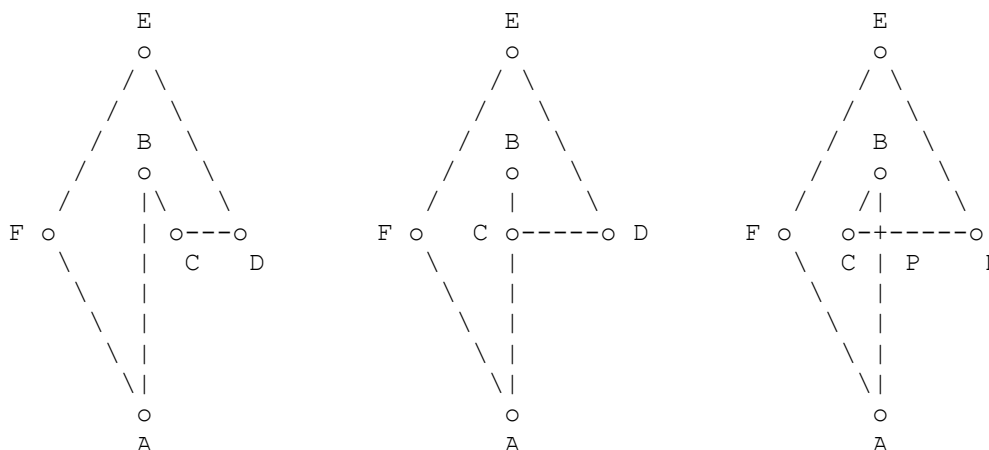
A real polytope is *monomorphic* when its realization (i.e., a mapping of the j-faces of an abstract polytope into entities of rank j in some Euclidean or non-Euclidean space) is pointwise one to one, i.e. its elements are pairwise disjoint. All spheroidal polytopes are monomorphic. A polytope that is not monomorphic is *polymorphic*. A polytope is monomorphic if and only if its elements are pairwise disjoint.

A star polytope is a polytope in which two nonadjacent facets have points in common or in which any of its facets is itself starry, i.e. star polytopes are polymorphic. The hexagon

8

just discussed shows that a polymorphic polytope is not necessarily starry. (The only two sides that intersect or overlap are the adjacent sides AB and BC.) A crossed quadrilateral is the simplest star polygon. A pentagram is the simplest regular star polygon.

The body of a star polygon generally consists of two or more distinct regions taken as a single entity. The body of a regular pentagram, for example, comprises a pentagonal core and five triangular extensions. This is still an open set but not an open region. Vertices in pairs are end points of (but do not belong to) the sides, and each vertex or side is part of the boundary of the body.

It may not be clear to everyone, but the hexagon should be seen as a transitional stage between the figures on the left and on the right below.

```
         E                        E                        E
         o                        o                        o
        / \                      / \                      / \
       /   \                    /   \                    /   \
      /  B  \                  /  B  \                  /  B  \
     /   o   \                /   o   \                /   o   \
    /   |\    \              /   |     \              /   /|    \
 F o    | o---o          F o  C o-----o D          F o   o-+-----o
    \   |  C  D              \   |                     \  C | P     D
     \  |                     \  |                      \   |
      \ |                      \ |                       \  |
       \|                       \|                        \ |
        o                        o                         \|
        A                        A                          o
                                                            A
```

In all three cases the vertices are the points A, B, C, D, E, and F, and the sides are the segments AB, BC, CD, DE, EF, and FA. The body of the first figure is a connected open region ABCDEF. The body of the second figure is a connected open region ACDEF with a slit from C to B. The body of the third figure is the union of two open regions, BCP and APCBPDEF. The point P is a "false vertex." In all three cases the boundary of the body is the union of the sides, the vertices, and the empty set. It is no more problematic that the second figure has an internal whisker than that the third one has a hidden side.
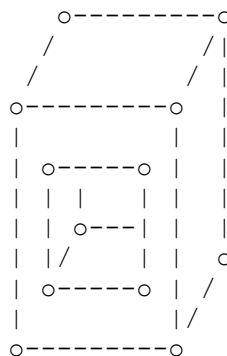
[*The allowing of slits but forbidding of whiskers prompted a lively discussion – Ed.*]

If we attached an inward-directed line before turning a sphere inside out a la Smale (see later), we would end up with a sphere with an outward-directed line. But so what? Smale's operation interchanges the outer and inner surfaces of the sphere, along with any finite attachments. It does not interchange the sphere's bounded interior and unbounded exterior. Indeed, it is the fact that we can tell the inside and the outside apart that makes the operation so remarkable. Far from being "pretty much the same," a bounded figure's inside and outside are fundamentally different. There is no reason that the same rules should apply to both.

Certain polyhedra such as semicupolas and sesquicupolas have membranes. Some categories of uniform polychora include pairs of isomorphic figures, one with a membrane, the other without. What characterizes a membrane is that part of the face does not touch the body of the polyhedron at all. My definition allows polyhedra with membranes, since it requires any face of a polyhedron to have a nonempty intersection with the closure of the body but not that every part of the face be in contact with the body.
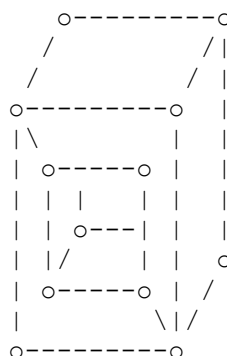
Another crucial provision in my definition of a real n-polytope is that, for $0 < j < n$, all the elements common to any pair of j-facials lie in a common (j-1)-space. This prevents two j-facials from being rigidly locked together by their common elements. To illustrate this

provision, as well as to emphasize another point, let me use a three-dimensional toroid as an example.

```
              o---------o
             /         /|
            /         / |
           o---------o  |
           |         |  |
           | o-----o |  |
           | | |   | | |
           | | o---| |  |
           | |/    | |  o
           | o-----o | /
           |         |/
           o---------o
```

The regions with holes are not proper entities, because their exteriors are not connected. So this is not a real polyhedron at all. It does not even qualify as an abstract polyhedron because it is not properly connected. It splits abstractly into a compound of two rectangular prisms; and compounds are not polytopes.
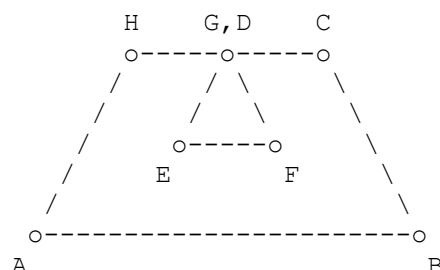
However, there is a remedy. Each of the illegal ring-shaped regions can be divided into two coplanar hexagonal regions, which *are* proper entities, by drawing two new edges:

```
              o---------o
             /         /|
            /         / |
           o---------o  |
           |\        |  |
           | o-----o |  |
           | | |   | | |
           | | o---| |  |
           | |/    | |  o
           | o-----o | /
           |       \|/
           o---------o
```

There is no prohibition against coplanar faces, but we must make sure that the two new edges dividing each ring-shaped region are collinear. This ensures that if the two new face polygons are taken by themselves the line of their common edges can function as a hinge; otherwise, they would be rigidly locked together in one plane. (If we had divided each ring-shaped region into three or four regions, this issue would not arise.) We now have a legitimate polyhedron, with 16 vertices, 28 edges, and 12 faces. Note that V - E + F = 0, which is the right value for a toroid of genus 1.

As another illustration of this requirement of "dyadic flexibility," consider the compound of a (small) icosahedron and a great dodecahedron with the same vertices and edges or the conjugate compound of a small stellated dodecahedron and a great icosahedron. These figures are sometimes called the "small complex icosidodecahedron" and the "great complex icosidodecahedron." There is no problem with either of these compounds, as they do not purport to be single polyhedra. But some of the figures in Jonathan Bowers' lists of uniform polychora have cell polyhedra that form such compounds. The vertices and edges common to a pair of polyhedra in one of these compounds do not all lie in a common plane, and the polyhedra are therefore rigidly locked together. Since they lack dyadic flexibility, I do not count such figures as real polychora.

As I have defined it, a "real" n-polytope is an abstract n-polytope whose j-faces are proper entities of rank j in some n-space and whose j-facials of lower rank are real j-polytopes. This requires the j-faces to be distinct (though not necessarily disjoint) entities. Thus a real polytope may not have coincident vertices, and while its edges may cross or even overlap, they may not coincide. The figure below demonstrates what can happen when the vertices of a polygon are allowed to coincide.

```
        H       G,D       C
        o-----o-----o
       /     / \      \
      /     /   \      \
     /     o-----o      \
    /     E       F      \
   /                      \
  o------------------------o
  A                        B
```

This figure, as a set of points, line segments, etc., can be regarded as the image in the Euclidean plane of an abstract 2-polytope <ABCDEFGH>. However, because 0-faces D and G are mapped into the same point, it is not a "faithful" image. In other words, the figure does not itself qualify as an abstract 2-polytope and so is not a real polygon. It can still be treated as a degenerate polygon. Aided by the labeling, we could deduce that there are 8 vertices (A, B, C, D, E, F, G, H) and 8 sides (AB, BC, CD, DE, EF, FG, GH, HA), with a body ABCDEFGH consisting of two polygonal regions. But, apart from alphabetical order, there is no way to tell whether, in making a circuit of the vertices and sides, we are supposed to visit vertex E or F first, i.e. how the loop DEF = EFG should be oriented. We could just as well see the body as a single region ABCDFEGH surrounding a hole.

Remarkably, no such ambiguity ever arises with polytopes satisfying the conditions in my definition. Once we have specified the individual entities, there is only one way they can be interpreted as the j-faces of a real polytope, and there can be no argument as to what polytope we are looking at. If the separate pieces are not entities or if there is no way to assemble them consistent with the definition, then whatever we have may be geometrically interesting, but it is not a real polytope. No real polytope ever needs to be labeled (though labeling may be helpful in discussing it).

My definition of a "real" polytope is broad enough to cover polytopes in both Euclidean and non-Euclidean (spherical, hyperbolic, or elliptic) spaces, to include both convex polytopes and star polytopes, to include both orientable and nonorientable figures, and to include both finite polytopes and Euclidean or hyperbolic apeirotopes. A slightly modified definition covers Euclidean and non-Euclidean honeycombs. This is not by any means the only way that one can define polytopes and honeycombs, but it does seem to be compatible with the kind of figures that people are generally interested in. Anyone should feel free to investigate figures whose properties do not satisfy my criteria, but in doing so one should be aware that not every imaginable polytope-like figure can be supported by a consistent theory.

Holes

[*We have seen that a toroidal polytope has external holes – Ed.*]

I now consider whether it makes sense to regard some polytopes as having *internal cavities* as "holes." The best answer is not at all obvious, and the question needs to be considered in the context of a satisfactory general definition of "polytope."

One aspect of the problem can be illustrated by the following pairs of polygons:
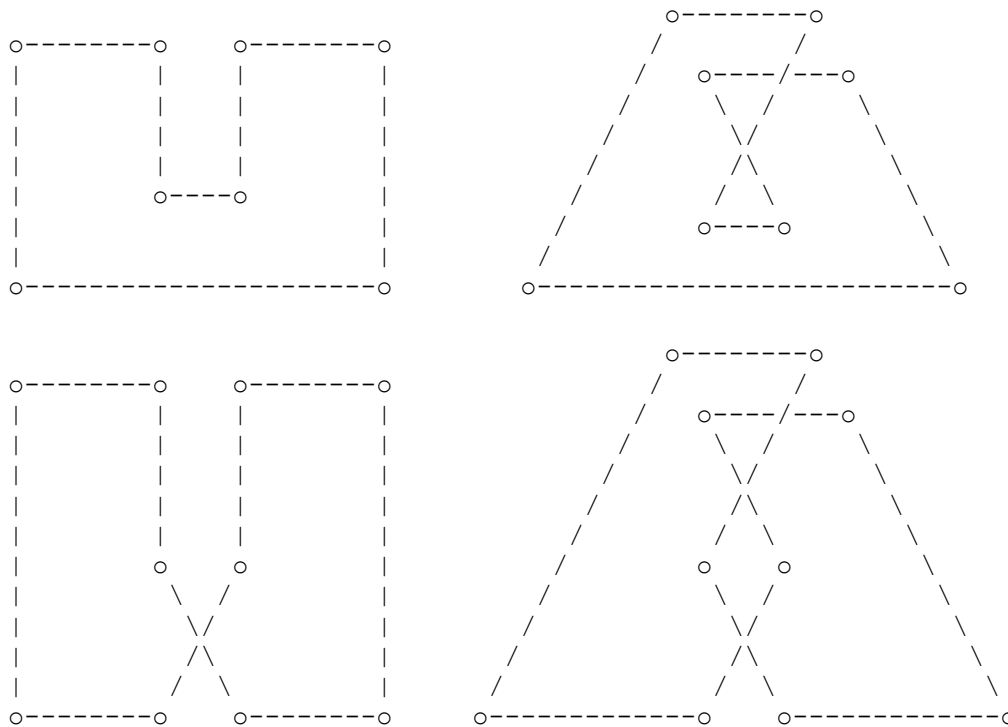
In each case the left-hand figure can be continuously deformed into the right-hand one. For the top pair, it seems plausible that the polygon on the right has a triangular hole that was formerly part of the exterior. For the bottom pair, the polygon on the right would likewise seem to have a rhombic hole. But what about the hidden triangular region? Should it be regarded as a doubly covered region or do the two overlapping parts cancel out, giving another hole?

At least polygons are orientable, so such questions make sense. But polyhedra and higher-dimensional polytopes can be nonorientable, making it unclear just what is inside or outside. And our intuition may not be reliable even with orientable figures.

If one were to form a prism with either of these star polygons as a base, then the "hole" would appear to be one end of a tunnel through a toroid. Now, unlike the holey polygon, the exterior of such a prism would be connected in three-space. Does this not show that the hole is significant after all? No, because the face polygons of a real polyhedron must be real polygons, and holey bases do not qualify. So we must fill in all the bounded regions of the star polygon, which then forces us to plug up the tunnel.

Stephen Smale famously showed that a sphere can be turned inside out, i.e., continuously deformed so that what was its outer surface becomes its inner surface and vice versa. If this can be done with a sphere, it could presumably be done with a spheroidal polyhedron having sufficiently many faces. And try to distinguish the solid parts from the "holes" at some of the intermediate stages!

Jonathan Bowers says that he doesn't have a precise definition of a hole yet but usually punches them out where it seems obvious. This leads him to identify twelve cavities or "holes" inside Groh, with a roundabout way of determining the density of its internal regions, even though this polyhedron is nonorientable. Perhaps we need a Potter Stewart rule for what counts as a hole ("I know it when I see it"), but I don't think we can ever formulate a consistent formal definition.

This brings me back to my original position, which is that all the enclosed regions bounded by the j-faces of an n-polytope (j < n) must be regarded as part of its body; there are no holes.

There is a close analogy between the problem of classifying enclosed regions created by the facets of a self-intersecting polytope and the problem of determining areas and volumes of bounded regions in the plane or in space. In each case some things that seem obviously true turn out to be false and what works in two dimensions turns out not to work in three.

There is a well-established theory of dissection of plane polygons. Any non-self-intersecting polygon can be dissected into a finite number of pieces that can be reassembled to form any other polygon of the same area. More generally, area of arbitrary regions is preserved under dissection and rearrangement. It has even been shown that it is possible to dissect a circular disk into a finite number of pieces that can be reassembled to form a square of the same area. The number of pieces required is quite large and some of the pieces have strange shapes, but it can be done.

The third in the famous list of problems posed by David Hilbert at the International Congress of Mathematicians at Paris in 1900 asked whether there exist "two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra." Within a year Max Dehn showed that a regular tetrahedron cannot be dissected into a finite number of pieces that can be reassembled to form a cube of the same volume. From this one readily obtains two tetrahedra of equal bases and altitudes that are not equidecomposable. So there is no dissection theory for polyhedra comparable to the one that works so nicely for polygons.

And that is not all. In 1924 Stefan Banach and Alfred Tarski proved what has come to be known as the Banach-Tarski Paradox. It is possible to dissect a solid sphere into six pieces that can be reassembled to form *two* solid spheres, each of the same volume as the original. Of course, the pieces are extremely complicated, so complicated that they do not even have volumes. Whereas areas of plane figures are preserved under all dissections and rearrangements, this is not the case with volumes of solid figures.

For an interesting account of these and other results, see Stewart (1992, Chapter 13).

There is likewise a well-established theory of the enclosed regions of the plane formed by an oriented polygon with no coincident vertices or overlapping edges. Each region can be assigned an integer value that is the difference between the number of left-pointing and right-pointing sides that one crosses in going from a point inside the region to a point completely outside the polygon.

Any oriented polygon can be continuously deformed into various other polygons. In the process angles and lengths of sides can change, provided that no two vertices ever coincide and no two sides ever overlap. All continuous deformations preserve the "winding number" of the polygon, the net number of counterclockwise revolutions made in traversing the vertices and sides in the direction of the orientation.

The invariance of the winding number makes it possible to classify the enclosed regions formed by oriented polygons. It is consistent with this classification to regard a region labelled '0' as belonging to the exterior of the polygon. In other words, some oriented polygons can be said to have "holes." By these lights, the right-hand figure of the first pair of polygons depicted in my initial message has one hole and the right-hand figure of the second pair has two.

A similar analysis, taking into account the fact that the core of a regular pentagram {5/2} is a region of density 2, allows one to show that the regular star polyhedra {5/2, 5} and {5, 5/2} have a core density of 3, while {5/2, 3} and {3, 5/2} have a core density of 7. The

density of any orientable uniform polyhedron none of whose faces pass through the center can be similarly calculated.

So it might appear that the theory that allows us to classify the enclosed regions of oriented star polygons could be extended to apply to star polyhedra and that in some cases enclosed regions should logically be regarded as holes. But in fact this cannot be done, for two reasons that I have previously alluded to. The first is that not all polytopes are orientable, and there is no consistent way to classify the regions formed by the facets of a nonorientable polytope. The second is that, as shown by the possibility of turning a sphere inside out, there is no property of an oriented polyhedron, invariant under continuous deformation, comparable to the winding number of an oriented polygon.

Since the idea of a polytopal "hole" cannot be successfully extended from the plane to spaces of three or more dimensions, there is not much point in trying to fit it into a general definition of "polytope." In fact, if we insist on doing so for self-intersecting figures other than polygons, the resulting theory will inevitably lead to contradictions.

My definition of a "real" polytope may not satisfy everyone's notions of what a polytope ought to be, but it takes in a lot of territory while still imposing a few reasonable restrictions. (At least I consider them reasonable; others may differ.) It avoids the danger of relying too much on one's intuition. Most important, for my purposes, it provides a basis for a consistent theory of uniform polytopes.

References

L. Danzer and E. Schulte
1982 "Regulare Inzidenzkomplexe, I," *Geom. Dedicata* 13, 295-308.

B. Grünbaum
1993 "Polyhedra with hollow faces," *Polytopes: Abstract, Convex, and Computational, Proc. NATO - ASI Conference, Toronto 1993,* T. Bisztriczky, P. McMullen, R. Schneider, and A. I. Weiss, eds. (Proc. NATO Adv. Study Inst., Scarborough, Ont., 1993; Kluwer, Dordrecht-Boston-London, 1994), 43-70.

P. McMullen
1989 "Realizations of regular polytopes," *Aequationes Math.* 37, 38-56.
1993 "Modern developments in regular polytopes," *Polytopes: Abstract, Convex, and Computational, Proc. NATO - ASI Conference, Toronto 1993,* T. Bisztriczky, P. McMullen, R. Schneider, and A. I. Weiss, eds. (Proc. NATO Adv. Study Inst., Scarborough, Ont., 1993; Kluwer, Dordrecht-Boston-London, 1994), 97-124.

P. McMullen and E. Schulte
2002 "Abstract Regular Polytopes," *Encyclopedia of Mathematics and Its Applications*, Vol. 92. Cambridge Univ. Press, Cambridge-New York, 2002.

I. Stewart
1992 *The Problems of Mathematics*, 2nd ed., Oxford Univ. Press, Oxford-New York, 1992.